The Plane Geometry of the Point in Point-Space of Four Dimensions.

By C. J. Keyser.

I.—*Introductory Considerations.*

1. As is well known, the dimensionality (in Riemann's sense) of any given space depends upon the element chosen for its construction; and in accordance with the Plücker principle of counting constants, any given space may be made to assume any prescribed dimensionality k by merely taking for element a configuration for whose determination within that space k independent data are necessary and sufficient—a configuration, in other words, whose general analytical representation in the given space involves exactly k parameters. A space being assumed, there are, in general, infinitely many possible choices of element for which the space will have a previously assigned dimensionality. Of such possible choices the great majority would be inexpedient as not leading to interesting results. Of all elements, in case of any given space, those are, in general, most practicable which present themselves in *pairs of reciprocals*, as in the familiar examples of the point and line in the plane, the line and plane in the sheaf, and the point and plane in ordinary space.

A space that is n-dimensional in points is also n-dimensional in point-spaces of $n - 1$ dimensions. It has $2(n - 1)$ dimensions both in lines and in point-spaces of $n - 2$ dimensions; and, in general, its dimensionality is $p(n - p + 1)$ if the point-space either of $p - 1$ or of $n - p$ dimensions be taken as element. Not only, however, do the two last mentioned elements furnish the same dimensionality, which is a necessary though not a sufficient condition for reciprocity, but they are indeed reciprocal elements in n-fold point-space; for the same system of equations, which on proper interpretation defines one of the elements, admits a second (dual) interpretation defining the other. It thus appears that by taking as elements the various point-spaces of less than n dimensions for the construc-

tion of n-fold point-space, there arise n geometries of this space; or, if we regard two reciprocal theories as but two phases of one geometry, the elements in question yield $\frac{n}{2}$ or $\frac{n-1}{2}+1$ geometries according as n is even or odd, the element having $\frac{n-1}{2}$ dimensions being, in case of n odd, its own reciprocal, or *self-reciprocal.*

Like considerations hold for spaces of n dimensions in other elements than points. It will be convenient, however, and sufficient to conduct this discussion for space supposed n-fold in points.

2. Of such geometries the self-reciprocal, or those arising from the use of self-reciprocal elements, are of special interest as well from the artistic as from the scientific point of view. The precise nature of the distinction in question may be made sufficiently clear by the following considerations. In n-fold space a definite configuration C, including this space itself as a special case, may, in general, be regarded at will as an assemblage of points or of lines or of planes and so on up to $\overline{n-1}$-fold point-spaces. These n assemblages, which may be denoted respectively by $E_0, E_1, \ldots, E_{n-1}$, the subscripts indicating the point dimensionality of the elements of the corresponding assemblages, are equivalent not only in the assemblage theory sense of the term but also in the logical sense that they serve as so many distinct definitions or conceptions of one and the same configuration. While distinct, they are of course not independent. If, for example, C be supposed to represent a curve of $\overline{n-1}$-ple curvature, E_0 will naturally be the assemblage of its points, E_1 the assemblage of its tangent lines, E_2 that of its osculating planes, ..., and the E's are accordingly to be thought as having a one-parameter dependence, by virtue of which to each element of C belonging to one E, there corresponds in general one and but one element of C belonging to each other E. Now, under a homographic transformation, the n E's are converted into n other assemblages $E'_0, E'_1, \ldots, E'_{n-1}$ in such a way that any E and the corresponding E' are of the same kind, have, i. e., the same subscript. The E''s are connected like the E's and in their turn serve as n distinct definitions of one and the same configuration C', the transformed of C. We may say, then, that each of the indicated definitions or conceptions of a given configuration is preserved in kind under a homographic transformation. Such is,

however, in general, not the case under a dualistic transformation; for, while the latter converts the n E's into n E''s, of which each serves as a definition of the transformed configuration C' of C, any E and the corresponding E' are, in general, not of a kind; if the subscript of the former be k, that of the latter will be $n - k - 1$, and these cannot be equal unless n be odd, and in this case only for a single value of k, namely, $k = \overline{n-1} : 2$, n being given. This case excepted, no definition of C is preserved under dualistic transformation; the point, line, ..., conceptions of C pass over respectively into the $\overline{n-1}$-space, $\overline{n-2}$-space, ..., conceptions of C'; but in the case where n is odd and equal (say) to $2m+1$, the assemblage E_m defining C is converted into an assemblage E'_m defining C'; the conception is preserved in kind. Now when $n = 2m+1$, the elements of E_m are self-reciprocal elements of n-fold space, and under no other circumstances are the elements of any E self-reciprocal. We arrive accordingly at this conclusion: *The distinction of the self-reciprocal geometries among other geometries is the definitional or conceptual invariance,* in case of the former, of all configurations, under both the homographic and the dualistic transformations.* Because of this property of invariance, one may say that the m-space conception of configurations in space of $2m+1$ dimensions is of higher scientific value, as being more central and penetrating, in more perfect accord with the intimate nature of space itself, than are such conceptions as lose their identity under one or the other of the mentioned modes of transformations—an estimate, moreover, that seems to be justified by the highly artistic analytical form which self-reciprocal theories are, it is well known, capable of assuming.

3. Point-space of 4 dimensions is also 4-dimensional in ordinary 3-dimensional spaces, or *lineoids*,† the point and the lineoid being reciprocal elements. It is 6-dimensional in lines and in planes, which are also reciprocal elements. This space contains no linear self-reciprocal element and admits of no self-reciprocal construction. Nevertheless there are two self-reciprocal theories of spaces (the point and the lineoid) *within* 4-fold point space, which, besides their own intrinsic

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* It is interesting that the significance of this property, which was pointed out and made a principle of procedure in the line geometry of ordinary space by Klein, Koenigs and others, seems not to have been fully appreciated by Plücker, whose method even in line theory never became quite free from the relatively cumbrous point-plane conception of space.

† Cf. Cole: "On Rotations in Space of Four Dimensions." Amer. Math. Journ., Vol. 12.

interest, are of the greatest importance in building up as well the point-lineoid as the line-plane geometry of 4-fold space itself. Just as any *lineoid* of this space is 3-dimensional in points and in planes and 4-dimensional in *lines*, so any *point* of the same space is 3-dimensional in lineoids and in lines and 4-dimensional in *planes;* and just as the line geometry (the Plücker theory) of a lineoid, regarded as a space of lines, is a self-reciprocal geometry, so the plane geometry of a point, regarded as a space or plenum of planes containing it, is a self-reciprocal theory. With the evidently possible parallelization of these coordinate self-reciprocal theories with the point-lineoid geometry of 4-space, we are not here concerned. Our interest lies in the theories as such, in their relations with one another and in the completely correlative rôles they play particularly in the development of the *line-plane* geometry of 4-space. The line geometry of the lineoid has been often treated and is familiar enough, at least in its elements. On the other hand, the plane geometry of the point (in 4-space) has not, so far as we are aware, been systematically developed.* This paper undertakes to construct so much of this theory as in connection with the other will be of immediate service in investigating the line-plane geometry of 4-space, to which subject this article is intended as a preliminary contribution.

II.—*Homogeneous Coordinates of the Plane.*

4. We enter here directly upon the subject proper of this paper: the plane theory of the point in 4-space. The space with which we have to deal is the point regarded as the assemblage of all the lineoids, planes and lines of 4-space that pass through (or contain) it. As the plane is to be taken as element, the point will be for us primarily a space of planes. This hypersheaf will be supposed given once for all, and, except where the contrary is indicated, all lines, planes and lineoids considered will be supposed to belong to it.

Let the assumed point be given by the four lineoids

$$A_1 = 0, \quad A_2 = 0, \quad A_3 = 0, \quad A_4 = 0,$$

where

$$A_k \equiv \alpha_5^{(k)} + \Sigma \alpha_i^{(k)} X_i \qquad (i = 1, 2, 3, 4).$$

The assemblage of generating lineoids may be represented by the equation

$$x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4 = 0,$$

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*In the cited article by Cole several theorems of the present paper are established.

where the x_i are parameters. Each system of values of x_i defines a lineoid and to each lineoid corresponds a unique system of values of the ratios $x_1 : x_2 : x_3 : x_4$ A linear equation

$$\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 = 0 \equiv \Sigma \xi_i x_i \tag{1}$$

where the ξ_i are supposed given, will define a *line* ξ_i as an *envelope* of *lineoids*. On the other hand if the ξ_i be regarded as variable and the x_i as given, the same equation will define a *lineoid* x_i as *locus* of *lines*, a bundle. Accordingly a pair of equations

$$\Sigma \xi_i x_i = 0, \quad \Sigma \xi_i' x_i = 0$$

will represent a *plane* (ξ_i, ξ_i') as an *envelope* of *lineoids*, while a pair

$$\Sigma x_i \xi_i = 0, \quad \Sigma x_i' \xi_i = 0$$

will represent a *plane* (x_i, x_i') as a *locus* of *lines*, a *flat pencil*.

We will employ x_i and ξ_i respectively as associated homogeneous lineoid and line coordinates.* The configuration of common reference will be that composed of the four lineoids A_1, A_2, A_3, A_4, the six planes A_1A_2, A_1A_3, A_1A_4, A_2A_3, A_2A_4, A_3A_4, and the four lines $A_1A_2A_3$, $A_1A_2A_4$, $A_1A_3A_4$, $A_2A_3A_4$. The coordinate lineoids will be represented in line coordinates by $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$, $\xi_4 = 0$, and in lineoid coordinates by $x_2 = x_3 = x_4 = 0$, $x_1 = x_3 = x_4 = 0$, $x_1 = x_2 = x_4 = 0$, $x_1 = x_2 = x_3 = 0$; the fundamental lines will be represented by $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, or by $\xi_2 = \xi_3 = \xi_4 = 0$, $\xi_1 = \xi_3 = \xi_4 = 0$, $\xi_1 = \xi_2 = \xi_4 = 0$, $\xi_1 = \xi_2 = \xi_3 = 0$; while the planes of reference will be given by $x_1 = x_2 = 0$, $x_1 = x_3 = 0$, $x_1 = x_4 = 0$, $x_2 = x_3 = 0$, $x_2 = x_4 = 0$, $x_3 = x_4 = 0$, or by $\xi_3 = \xi_4 = 0$, $\xi_2 = \xi_4 = 0$, $\xi_2 = \xi_3 = 0$, $\xi_1 = \xi_4 = 0$, $\xi_1 = \xi_3 = 0$, $\xi_1 = \xi_2 = 0$. The equation (1) also signifies that the line ξ_i and the lineoid x_i are *united in position*, *i. e.*, that the *line lies in the lineoid* and the *lineoid contains the line.*

5. As already indicated, any plane whatever may appear in either of two aspects: as a *locus* of its *lines*, *i. e.*, as a *flat pencil*, or as an *envelope* of its generating *lineoids*, *i. e.*, the lineoids containing the plane. These dual conceptions of the plane correspond precisely, in the order named, to the two Plücker

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*Such systems might legitimately have been assumed immediately by virt. e of the projective correspondence, already noticed, between the elements of the hypersheaf under investigation and the elements of the lineoid, according to which the lines, planes and lineoids of the one assemblage correspond respectively in a one-to-one way to the planes, lines and points of the other.

conceptions of the line (in ordinary space), namely, axis and ray (*Axe*, *Strahl*). The plane, being geometrically determined as a flat pencil by any two of its lines, is determined analytically by two sets of line coordinates; while, being geometrically determined as an envelope by any two of its (generating) lineoids, it is determined analytically by two systems of lineoid coordinates. As explained below, the first two sets when combined will furnish one system of homogeneous plane coordinates and the second two sets similarly combined will yield a second system.

Consider any two lines ξ_i and η_i $(i = 1, 2, 3, 4)$. These determine a plane π, which is equally determined by any two lines of the pencil

$$\Sigma(\lambda\xi_i x_i + \mu\eta_i x_i) = 0, \qquad (i = 1, 2, 3, 4), \tag{2}$$

and in particular by any two of the four special lines obtained by equating successively to zero the coefficients of x_1, x_2, x_3, x_4 in (2),

$$\left.\begin{aligned} (\xi_1\eta_2 - \xi_2\eta_1)x_2 + (\xi_1\eta_3 - \xi_3\eta_1)x_3 + (\xi_1\eta_4 - \xi_4\eta_1)x_4 = 0,\\ -(\xi_1\eta_2 - \xi_2\eta_1)x_1 + (\xi_2\eta_3 - \xi_3\eta_2)x_3 + (\xi_2\eta_4 - \xi_4\eta_2)x_4 = 0,\\ -(\xi_1\eta_3 - \xi_3\eta_1)x_1 - (\xi_2\eta_3 - \xi_3\eta_2)x_2 + (\xi_3\eta_4 - \xi_4\eta_3)x_4 = 0,\\ -(\xi_1\eta_4 - \xi_4\eta_1)x_1 - (\xi_2\eta_4 - \xi_4\eta_2)x_2 - (\xi_3\eta_4 - \xi_3\eta_4)x_3 = 0. \end{aligned}\right\} \tag{3}$$

These are the four lines of the pencil (2) *that lie one in each of the fundamental lineoids*. The ratios of the coefficients of any two of the lines furnish the four constants upon which the position of π depends. The choice of two of the lines would, however, be arbitrary, and we may avoid such choice, while at the same time securing symmetry, by retaining for coordinates of π the entire six coefficients of equations (3). These, again, may be replaced by an arbitrary multiple of them, since only their ratios are material. We will accordingly have for *coordinates of the plane regarded as a flat pencil* the six quantities

$$\rho p_{ik} = \xi_i\eta_k - \xi_k\eta_i. \tag{4}$$

On expanding the identically vanishing determinant

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{vmatrix} = \Delta,$$

in terms of quadratic minors, we find that the six coordinates are connected

by the identity

$$\tfrac{1}{2}\omega(p) \equiv 0 = p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23}, \tag{5}$$

showing, as ought to be the case, that the 5 ratios of the 6 p's are equivalent to but 4 independents. It can be readily shown that any six quantities p_{ik} satisfying the identity (5) and being such that $p_{ik} = -p_{ki}$, serve to determine a plane (or pencil), and that if ξ_i' and η_i' be any two lines of the pencil, $p_{ik} : p_{ik}' = k$, a constant.

5. To find a corresponding *system of coordinates for the plane conceived as an envelope of lineoids,* suppose it given by two lineoids x_i and y_i. Of the lineoids of the pencil

$$\Sigma\,(\lambda x_i\,\xi_i + \mu y_i\,\xi_i) = 0, \qquad (i = 1,\ 2,\ 3,\ 4) \tag{6}$$

of generators of π, the following

$$\begin{cases} q_{12}\,\xi_2 + q_{13}\,\xi_3 + q_{14}\,\xi_4 = 0, \\ q_{21}\,\xi_1 + q_{23}\,\xi_3 + q_{24}\,\xi_4 = 0, \\ q_{31}\,\xi_1 + q_{32}\,\xi_2 + q_{34}\,\xi_4 = 0, \\ q_{41}\,\xi_1 + q_{42}\,\xi_2 + q_{43}\,\xi_3 = 0, \end{cases}$$

where

$$\sigma q_{ik} = x_i y_k - x_k y_i \tag{8}$$

are the four generators of which each contains one and but one of the fundamental lines.

The six coefficients q_{ik} are connected by the quadratic identity

$$\tfrac{1}{2}\,\omega\,(q) \equiv 0 = q_{12}\,q_{34} + q_{13}\,q_{42} + q_{14}\,q_{23}, \tag{9}$$

and for reasons precisely analogous to those given for the p's, any six quantities q_{ik} satisfying the identity (9) and being such that $q_{ik} = -q_{ki}$, suffice to determine a plane uniquely, and may be taken for homogeneous coordinates of the same regarded as enveloped by lineoids.

7. Inasmuch as the configurations with which we shall be concerned are, most of them, to be conceived as assemblages of *planes*, and since the latter are self-reciprocal elements (cf. §I), there is, in general, no advantage, but often rather a disadvantage, in observing the distinction between the two aspects of the plane, as flat pencil of lines and as envelope of lineoids, in which alone the

difference between the two systems* p_{ik} and q_{ik} originates. And in fact it is easy to show that these systems, while they differ in the sense indicated, are *as* coordinates, as data fixing the position of a plane, *identical* function for function, a proportionality factor being of course excepted. To effect this identification it is sufficient to find the condition that p_{ik} and q_{ik} shall determine one and the same plane. Suppose the plane determined by p_{ik} to be that represented by equations (3). If q_{ik} give the same plane, then the latter must lie in each of the lineoids x_i, y_i, which requires

$$\left.\begin{aligned} p_{12}x_2 + p_{13}x_3 + p_{14}x_4 = 0,\\ p_{12}y_2 + p_{13}y_3 + p_{14}y_4 = 0,\end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} p_{21}x_1 + p_{23}x_3 + p_{24}x_4 = 0,\\ p_{21}y_1 + p_{23}y_3 + p_{24}y_4 = 0,\end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} p_{31}x_1 + p_{32}x_2 + p_{34}x_4 = 0,\\ p_{31}y_1 + p_{32}y_2 + p_{34}y_4 = 0,\end{aligned}\right\} \tag{12}$$

and a third pair, not needed.

From (10) we derive

$$p_{12} : q_{34} = p_{13} : q_{42} = p_{14} : q_{23},$$

from (11)

$$p_{12} : q_{34} = p_{23} : q_{14} = p_{42} : q_{13},$$

and from (12)

$$p_{13} : q_{42} = p_{23} : q_{14} = p_{34} : q_{12}.$$

On combining we have

$$p_{12} : q_{34} = p_{13} : q_{42} = p_{14} : q_{23} = p_{34} : q_{12} = p_{42} : q_{13} = p_{23} : q_{14}, \tag{13}$$ †

which shows that if the quantities p_{ik}, taken in any order, as 12, 13, 14, 34, 42, 23, determine a plane as a flat pencil, the same quantities, taken in the equally general corresponding order 34, 42, 23, 12, 13, 14, determine the same plane as an envelope of lineoids.

8. In general, two planes have no intersection, *i. e.*, no common line. The condition that they shall have a line in common, or, what is tantamount, shall

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* There are, of course, many equivalent systems of coordinates for the plane in the hypersheaf as there are for the line in ordinary space. Indeed, in the appendix to his very first paper on the latter subject Plücker presents no less than eight distinct systems. Cf. Plücker, "On a New Geometry of Space." Phil. Trans. of the R. Soc. of London, Vol. 155. Also Wiss. Abh.

† Cf. Plücker: "Neue Geometrie des Raumes," p. 4.

lie in a same lineoid, will, like the corresponding condition* for the intersection of two lines in the Plücker line theory, assume the following four forms, if the distinction of locus and envelope be observed. According as the two planes π and π' are regarded (*a*) both as flat pencils, (*b*) π as a pencil and π' as an envelope, (*c*) π as an envelope and π' as a pencil, (*d*) both as envelopes, the forms in question will be

$$p_{12}p'_{34} + p_{13}p'_{42} + p_{14}p'_{23} + p_{34}p'_{12} + p_{42}p'_{13} + p_{23}p'_{14} = 0, \qquad (a)$$

$$\Sigma p_{ik}\cdot q'_{ik} = 0, \quad \Sigma p'_{ik}\cdot q_{ik} = 0, \qquad (b), (c)$$

$$q_{12}q'_{34} + q_{13}q'_{42} + q_{14}\cdot q'_{23} + q_{34}\cdot q'_{12} + q_{42}\cdot q'_{13} + q_{23}\cdot q'_{14} = 0. \qquad (d)$$

By virtue, however, of (13) we may write for coordinates of the plane simply six quantities r_{ik}, such that $r_{ik} = -r_{ki}$ and that $\omega(r) = 0$, and then disregard the distinction of locus and envelope. The condition that two planes

$$\left\{\begin{aligned} r_{12}v_2 + r_{13}v_3 + r_{14}v_4 &= 0, \\ -r_{12}v_1 + r_{23}v_3 + r_{24}v_4 &= 0, \end{aligned}\right.$$

$$\left\{\begin{aligned} r'_{12}v_2 + r'_{13}v_3 + r'_{14}v_4 &= 0, \\ -r'_{12}v_1 + r'_{23}v_3 + r'_{24}v_4 &= 0, \end{aligned}\right.$$

shall lie in a same lineoid, or intersect in a line, is, then,

$$\begin{vmatrix} 0 & r_{12} & r_{13} & r_{14} \\ r_{12} & 0 & r_{23} & r_{24} \\ 0 & r'_{12} & r'_{13} & r'_{14} \\ r'_{12} & 0 & r'_{23} & r'_{24} \end{vmatrix} = 0,$$

from which by help of the conditions, $r_{ik} = -r_{ki}$ and $\omega(r) = 0$, we readily find

$$r_{12}r'_{34} + r_{13}r'_{42} + r_{14}r'_{23} + r_{34}r'_{12} + r_{42}r'_{13} + r_{23}r'_{14} = 0.$$

Writing the left member of this polar form

$$\omega(r, r')\dagger \equiv \tfrac{1}{2}\Sigma\frac{\partial\omega(r)}{\partial r_{ik}}r'_{ik} = \tfrac{1}{2}\Sigma\frac{\partial\omega(r')}{\partial r'_{ik}}r_{ik},$$

we have the fundamental proposition: *The necessary and sufficient condition that*

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* Cf. Cayley: "On the Six Coordinates of a Line," Collected Papers, Vol. VII. Klein: "Einleitung in die höhere Geometrie," Vol. 1, p. 168.

† Cf. Pasch: "Zur Theorie der linearen Complexe," Crelle, Vol. 75, p. 11. Also, Koenigs: "La geometrie reglé," Annales de La Faculte des Sciences de Toulouse, Vol. III, p. 9.

two planes r_{ik} and r'_{ik} shall have a line in common, or lie in a same lineoid, is that the polar form $\omega(r, r')$ with respect to these planes, shall vanish.

9. The coordinates r_{ik} admit of generalization. We know from the theory* of forms that the new variables ν_i in the transformation

$$\rho r_{ik} = C_{ik,1}\nu_1 + C_{ik,2}\nu_2 + \dots + C_{ik,6}\nu_6,$$

where the modulus is not zero, are connected by a homogeneous quadratic identity $\Omega(\nu) = 0$, where $\Omega(\nu)$ is the transformed of $\omega(r)$. Moreover, the polar form $\omega(r, r')$ of $\omega(r)$ is converted by the same transformation into the polar form $\Omega(\nu, \nu')$ of $\Omega(\nu)$. It is well known that $\omega(r)$ regarded as a quadratic form has a non-vanishing discriminant and that it is possible to find a linear transformation which will convert this form into any quadratic form $\Omega(\nu)$ whose determinant does not vanish. Accordingly we may employ for homogeneous plane coordinates any six variables ν_i connected by the quadratic relation $\Omega(\nu) = 0$, where $\Omega(\nu)$ has a non-zero discriminant. Hence the condition that the planes ν_i and ν'_i shall intersect in a line, or lie in one lineoid, is that the polar form $\Omega(\nu, \nu')$ shall vanish.

10. It is now perfectly clear, it was indeed *a priori* evident, that the theory here in process of construction and the line theory of ordinary space, while they are geometrically distinct, disparate in fact, may be made to assume one and the same analytical aspect. Accordingly three courses lie open. The theories being coordinate in rank and being correlative auxiliary instruments for the construction of the line-plane geometry of 4-space, the ideal would seem to be to develop them as such, side by side. On the other hand, as the line theory already exists in a score of presentations, one might be content to derive the plane theory from it by translation, by merely replacing the old system of ideas by the new. Again, as neither doctrine can claim logical priority as against the other, it appears to be desirable to present the new doctrine *once* on its own account, the old having been often so presented, and not as a secondary discipline derived from another. The first course is rejected as being too long; the second is scarcely shorter and offers, besides, a false perspective. The third recommends itself as a compromise, and accordingly we shall continue, as we have begun, to construct the theory in question, as self-justified, upon its own foundations,—a course which will allow occasional pauses to note correlative propositions in the corresponding line geometry.

* Cf. Klein: Op. cit., pp. 190, 191.

III.—*Systems of Planes.—The Linear Complex of Planes.*

11. We pass to the study of systems of planes. Of such systems there are five sorts as follows: (*a*) the *4-parameter* system, which is composed of all the planes of the point, or hypersheaf under investigation, and which may be regarded as the locus of a single plane π of the system, π being subject to no condition; (*b*) the *3-parameter* system, or *complex*, which is defined by imposing *one* condition upon the 4-parameter system; (*c*) the *2-parameter* system, or *congruence*, the assemblage defined by a pair of conditions upon the planes of the hypersheaf; (*d*) the *1-parameter* system, or *configuration* or *plane series*, an assemblage defined by a 3-fold condition; (*e*) the *zero-parameter* system, always a finite assemblage, defined by a set of *four* conditions upon the parameters of system (*a*). A plane will be said to have 4, 3, 2, 1, or 0 degrees of freedom or indetermination according as it is regarded as belonging to a 4-, 3-, 2-, 1-, or 0-parameter system.

12. Two or more planes having a line in common may be called *collinear;* two or more planes contained in a same lineoid may be called *collineoidal.* An assemblage of planes that are all of them at once collinear and collineoidal is an ordinary *axal pencil* of planes. We will, however, call such a pencil a *flat axal pencil,* reserving the name *axal pencil* for the totality of planes containing a line. Denote by ν_i' and ν_i'' any two collinear, or collineoidal, planes and consider the expression

$$\nu_i = \lambda_1 \nu_i' + \lambda_2 \nu_i''. \qquad \text{(the } \lambda\text{'s arbitrary)}$$

We have by hypothesis

$$\begin{cases} \Omega(\nu') = 0, \\ \Omega(\nu'') = 0, \\ \Omega(\nu', \nu'') = 0. \end{cases}$$

Also, by identity

$$\Omega(\nu) = \Omega(\lambda_1 \nu' + \lambda_2 \nu'') = \Omega(\nu')\lambda_1^2 + \Omega(\nu'')\lambda_2^2 + 2\Omega(\nu', \nu'')\lambda_1\lambda_2,$$

whence

$$\Omega(\nu) = 0,$$

i. e., the quantities ν_i are the coordinates of a plane for all values of λ_1 and λ_2. If π_i be any plane whatever having a lineoid in common with each of the planes ν_i' and ν_i'',

$$\Omega(\nu', \pi) = 0, \quad \Omega(\nu'', \pi) = 0,$$

and therefore

$$\Omega(\nu, \pi) \equiv \Omega(\nu', \pi)\lambda_1 + \Omega(\nu'', \pi)\lambda_2 = 0,$$

i. e., the planes ν_i are collineoidal with π_i and they consequently contain the common line of ν_i' and ν_i''. It is likewise plain that the planes ν_i are all contained in the common lineoid of ν_i' and ν_i''. The planes ν_i are therefore all found in the flat axal pencil (ν_i', ν_i''). Is the converse true? Is every plane of the pencil one of the planes ν_i? Suppose ν_i''' to be an arbitrarily chosen plane of the pencil and let π_i' be any plane collineoidal with ν_i''' but not with any other plane of the pencil. Let ν_i'''' be that one of the planes ν_i for which

$$\Omega(\nu, \pi') = \Omega(\nu', \pi')\lambda_1 + \Omega(\nu'', \pi')\lambda_2 = 0.$$

The planes ν_i''' and ν_i'''' are identical. We see, therefore, that the planes ν_i constitute the flat axal pencil (ν_i', ν_i''). Accordingly, any two *collineoidal planes ν_i' and ν_i'' determine a flat axal pencil and the coordinates of the planes of the pencil are of the form*

$$\nu_i = \lambda_1 \nu_i' + \lambda_2 \nu_i''.$$

This last is identical with the form giving in ordinary space the line coordinates of the lines of a flat pencil determined by two concurrent lines.

13. As a plane of a flat axal pencil has one degree of freedom and that of a complex three degrees, a plane that belongs to both will have zero degrees of freedom, being subject to four conditions. The number of planes of a given complex that belong to an arbitrary flat axal pencil is, therefore, finite. This number will be called the *degree* of the given complex.

The assemblage of planes having a line in common—the axal pencil proper—and the assemblage of planes contained in a lineoid—the ordinary bundle of planes—are the analogues respectively of the sheaf and the plane of lines in ordinary space. The two assemblages in question, *i. e.*, the axal pencil and the bundle, being each bi-dimensional, may with propriety receive a common name. Following a suggestion of Koenigs, we will adopt for such common designation the term *hyperpencil* of planes.

14. We may now prove that a *hyperpencil of planes is completely determined by any three planes ν_i', ν_i'', ν_i''', such that each is collineoidal* (*or collinear*) *with each of the other two, and that all and only the planes of the hyperpencil are given by coordinates of the form*

$$\nu_i = \lambda_1 \nu_i' + \lambda_2 \nu_i'' + \lambda_3 \nu_i'''.$$

Two cases may arise. The three given planes may determine three distinct lines and one lineoid containing them or three distinct lineoids and one line contained in them. In the former case the planes are the faces of an ordinary trieder and the hyperpencil will be a bundle. We will conduct the argument for the second case, for which the hyperpencil will be an axal pencil, the proof being identical in form for both cases. By hypothesis, we have

$$\Omega(\nu') = 0, \quad \Omega(\nu'') = 0, \quad \Omega(\nu''') = 0,$$
$$\Omega(\nu'', \nu''') = 0, \quad \Omega(\nu''', \nu') = 0, \quad \Omega(\nu', \nu'') = 0,$$

from which it follows that

$$\begin{aligned}\Omega(\nu) &\equiv \Omega(\lambda_1 \nu' + \lambda_2 \nu'' + \lambda_3 \nu''') \\ &\equiv \Omega(\nu')\lambda_1^2 + \Omega(\nu'')\lambda_2^2 + \Omega(\nu''')\lambda_3^2 + 2\Omega(\nu'', \nu''')\lambda_2\lambda_3 \\ &\qquad + 2\Omega(\nu''', \nu')\lambda_3\lambda_1 + 2\Omega(\nu', \nu'')\lambda_1\lambda_2 = 0.\end{aligned}$$

Hence for every system of values of the ratios $\lambda_1 : \lambda_2 : \lambda_3$, the six quantities ν_i determine a plane. Now let π_i be an arbitrary plane collineoidal (or collinear) with each of the given planes ν_i', ν_i'', ν_i'''. Then

$$\Omega(\pi) = 0, \quad \Omega(\nu', \pi) = 0, \quad \Omega(\nu'', \pi) = 0, \quad \Omega(\nu''', \pi) = 0.$$

Consequently

$$\Omega(\nu, \pi) \equiv \Omega(\lambda_1 \nu' + \lambda_2 \nu'' + \lambda_3 \nu''') \equiv \Omega(\nu', \pi)\lambda_1 + \Omega(\nu'', \pi)\lambda_2 + \Omega(\nu''', \pi)\lambda_3 = 0,$$

i. e., every ν_i is collineoidal with every π_i and hence contains the line (ν', ν'', ν'''). Conversely, every plane ν_i''' containing this line is one of the planes ν_i. For let π_i' and π_i'' be any two planes each collineoidal with but not belonging to the axal pencil. Only one plane ν_i''' is collineoidal with each of the planes π_i' and π_i''. We prove that one of the planes ν_i is so collineoidal, whence it follows that this ν_i is identical with ν_i''''. The proof consists in showing that

$$\Omega(\nu, \pi') = 0, \quad \Omega(\nu, \pi'') = 0.$$

Now

$$\Omega(\nu, \pi') = \Omega(\nu', \pi')\lambda_1 + \Omega(\nu'', \pi')\lambda_2 + \Omega(\nu''', \pi')\lambda_3,$$
$$\Omega(\nu, \pi'') = \Omega(\nu', \pi'')\lambda_1 + \Omega(\nu'', \pi'')\lambda_2 + \Omega(\nu''', \pi'')\lambda_3,$$

which may both be made to vanish by a proper choice of values of the ratios of the λ's. Hence the planes ν_i constitute the planes of the axal pencil determined by the collinear planes ν_i', ν_i'', ν_i'''. In like manner, if ν_i', ν_i'', ν_i''' be the

faces of an ordinary trieder, they determine a bundle whose planes are given by the formula

$$\nu_i = \lambda_1 \nu_i' + \lambda_2 \nu_i'' + \lambda_3 \nu_i'''.$$

15. A plane that is required to belong at once to a complex and a hyperpencil, being subject to three conditions, has one degree of freedom. The locus of such a plane is, therefore, a "configuration." We will name it a cone C_1, or a cone C_2 of the complex according as the hyperpencil is an axal pencil or a bundle. C_1 and C_2 correspond precisely and respectively to the curve and the cone of a complex of lines in ordinary space. Just as the curve has all its lines in a bi-dimensional *point* manifold, the *plane*, so C_1 has its planes in a bi-dimensional *lineoid* manifold, the *line;* and just as the (line) cone has all its lines joined by a point while their points require for their representation a 3-fold manifold of points, *ordinary space* (a *lineoid*), so C_2 has all its planes in a lineoid while their (generating) lineoids require for their construction a 3-fold manifold of lineoids, a *point;* and so on. Every line has its C_1 and every lineoid its C_2 of any given complex. A flat axal pencil will be said to belong to a given hyperpencil when the latter contains the planes of the former. The *degree* of a C_1 or a C_2 will signify the number of planes common to the cone and an arbitrary flat axal pencil belonging to the hyperpencil to which the cone belongs. It should be noted that as the notions, *locus* and *envelope*, of the Plücker geometry correspond respectively to *envelope* and *locus* in the present theory, so also the notions of *order* and *class* in the former doctrine correspond to those of *class* and *order* in the latter. Thus the curve of a line complex is an *envelope* of *lines*, but its correlate, C_1 of a plane complex, is a *locus* of *planes*. The degree of C_1 will be called the *order* of this cone, and the degree of C_2 will be called its *class*. We have immediately the proposition: *The degree of a complex is equal to the order of any of its C_1's and to the class of any of its C_2's.*

16. A complex of first degree is said to be *linear*. A C_1 of such a complex is of order 1, it is a flat axal pencil, to be viewed as a lineoid of collinear planes; while a C_2 of the linear complex, being of class 1, is also a flat axal pencil, to be viewed, however, as a line of collineoidal planes: the lineoid C_1 is a *locus* of the planes of the flat axal pencil; the line C_2 is an *envelope* of the planes of the flat axal pencil: the line and the lineoid are thus but reciprocal phases of one con-

figuration, just as in line geometry the flat pencil is regarded now as a point and again as a plane. Given an arbitrary linear complex of planes. Of these there pass through any line whatever a single infinity of planes all contained in a lineoid and constituting a flat axal pencil; the lineoid so determined will be called the *polar* lineoid of the given line. Reciprocally every lineoid contains a single infinity of the planes of the given complex and these, too, are collinear, constituting a flat axal pencil; the axis, or line so determined, will be called the *polar* line of the given lineoid. Accordingly *with respect to any linear plane complex, every lineoid has a polar line, and every line has a polar lineoid.* Every lineoid or line is *united in position* with its polar line or lineoid.

17. These propositions, showing the distribution of the planes of a linear complex, are of such fundamental importance as to justify their separate establishment by analytical means. As a preliminary we will show that a linear plane complex is representable by an equation of first degree in ν_i, and conversely, that every such equation defines such a complex.

Let the equation

$$F(\nu_i) = 0$$

represent a linear complex of planes. The identity

$$\Omega(\nu) = 0$$

is, of course, supposed given. Denote by ν_i' and ν_i'' any two collineoidal planes. We have seen that the coordinates of the planes of the flat axal pencil determined by ν_i' and ν_i'' are

$$\nu_i = \lambda_1 \nu_i' + \lambda_2 \nu_i''.$$

The condition that one of these planes shall belong to F is

$$F(\lambda_1 \nu_i' + \lambda_2 \nu_i'') = 0.$$

Since by definition of F, only one plane of the pencil belongs to F, the last equation must be linear in $\lambda_1 : \lambda_2$, and is, therefore, of the form

$$\Sigma c_i \nu_i = 0.$$

The converse is obviously correct.

Now let π denote any plane whatever and let (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4)

be any two generating lineoids of π. Then for coordinates of π we may take

$$\begin{cases} \nu_1 = x_1 y_2 - x_2 y_1, & \nu_4 = x_2 y_3 - x_3 y_2, \\ \nu_2 = x_1 y_3 - x_3 y_1, & \nu_5 = x_2 y_4 - x_4 y_2, \\ \nu_3 = x_1 y_4 - x_4 y_1, & \nu_6 = x_3 y_4 - x_4 y_3. \end{cases}$$

The condition that π shall belong to the complex

$$\Sigma c_i \nu_i = 0,$$

may, therefore, be written

$$(c_1 y_2 + c_2 y_3 + c_3 y_4) x_1 + (-c_1 y_1 + c_4 y_3 + c_5 y_4) x_2 \\ + (-c_2 y_1 - c_4 y_2 + c_6 y_4) x_3 + (-c_3 y_1 - c_5 y_2 - c_6 y_3) x_4 = 0.$$

This equation, if the y's be regarded as fixed and the x's as variable represents a straight line, and as the equation is satisfied by $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, $x_4 = y_4$, this line lies in the lineoid y. It thus appears that the *planes π of a given lineoid y that belong to a given linear complex envelope a line, the polar of a given lineoid. They constitute a flat axal pencil within y.*

In like manner, if π be supposed given by two of its lines ξ and η, reasoning analogous to the foregoing will show that the *planes of a given line y that belong to a given linear complex have for locus a lineoid, polar of the given line.*

The flat axal pencils which are thus determined, one for each line and one for each lineoid, by any given linear complex, may be called the *pencils of the complex.*

Denote by l any line and by L any lineoid containing l, and consider L' and l', the polars respectively of l and L with respect to a given linear plane complex C. The plane (L, L') being contained in L' and containing the polar l of L', belongs to C, and, therefore, as it is contained in L, it contains l'. Consequently, l' lies in L'. Hence, *the polars of a line and lineoid united in position are themselves united in position.*

Let π be any plane. Every generating lineoid L of π is united in position with every generating line l of π. Hence, every polar line l' of the L's is united in position with every polar lineoid L' of the l's Hence, the L''s and the l''s generate one and the same plane π'. Two planes π and π' thus related will be called *conjugate planes with respect to the given complex. Two conjugate planes are such that either of them is the locus* (*or envelope*) *of the polar lines* (*or lineoids*) *of the generating lineoids* (*or lines*) *of the other.*

18. It thus appears that a linear complex of planes serves as a dualistic transformation establishing a unique and reciprocal correspondence* between lines and lineoids, and between planes and planes. In this correspondence each plane of the complex corresponds to itself; for obviously, if π belongs to the complex, π and its conjugate π' coincide, i. e., *every plane of the given complex is self-conjugate or self-polar with respect to that complex.* On the other hand, no other plane is self-conjugate. In fact, if two conjugates π and π' are not planes of the complex, they are not collineoidal, for suppose them contained in a lineoid L; the polar line of L lies in both π and π', and hence these planes belong to the complex and consequently coincide. Therefore, *two conjugate planes coincide and so belong to the complex or else they are non-collinear and so do not belong to the complex.* This proposition is a corollary to the following: *If two conjugates, π_1 and π_1', are each collineoidal with the plane π, the latter belongs to the complex.* To prove this proposition, denote by L_1 the lineoid determined by π_1 and π, and by L_1' that determined by π_1' and π; the polar line l_1 of L_1 lies in π_1', and hence in L_1', and the polar line l_1' of L_1' lies in π_1 and hence in L_1; therefore, l_1 and l_1' are both lines of π, the common plane of L_1 and L_1'; hence π belongs to the complex.

Let π, any plane of the complex, be collineoidal with a plane π_1. If L be the lineoid containing π and π_1, the polar line l of L lies in π, and as L contains π_1, l also lies in π_1', the conjugate of π_1'; hence, *if a plane of a complex is collineoidal with any other plane, it is also collineoidal with the conjugate of the latter.*

19. The foregoing and additional properties of conjugate planes may be investigated analytically as follows: The condition

$$\Omega(\nu', \nu) = 0 = \Sigma \frac{\partial \Omega(\nu')}{\partial \nu_i'} \nu_i$$

that the planes ν_i' and ν_i shall be collineoidal (or collinear) will assume the form

$$\nu_4'\nu_1 + \nu_5'\nu_2 + \nu_6'\nu_3 + \nu_1'\nu_4 + \nu_2'\nu_5 + \nu_3'\nu_6 = 0$$

on taking $\Omega(\nu)$ to be of the form

$$\nu_1\nu_4 + \nu_2\nu_5 + \nu_3\nu_6.$$

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* Exceptions to the one-to-one character of this correspondence will be noted at a later stage.

The condition, being linear in ν_i' and ν_i, shows that the assemblage of planes of which each is collineoidal with a given plane is a linear complex. Such a linear complex will be called a *special* complex. The condition that the complex

$$\Sigma c_i \nu_i = 0$$

shall be a special complex is thus seen to be

$$\Omega(c) = c_1 c_4 + c_2 c_5 + c_3 c_6 = 0,$$

which is identical with the condition in the Plücker geometry that every line of a line complex shall have a point in common with a given line. Employing Klein's terminology for the line theory, we will call the quadratic form $\Omega(c)$ the *invariant* of the complex. In case of a special complex the plane which is collineoidal with each plane of the complex will be called the *director* plane or *directrix* of the complex, this plane being the analogue of the directrix of the special line complex of the line theory.

Let

$$\Sigma c_i x_i = 0 \tag{1}$$

be an arbitrary chosen complex, and denote by ν_i' any given plane. The latter is director plane of the special complex

$$\Omega(\nu', \nu) = 0. \tag{2}$$

The planes common to (1) and (2) are identical with the planes of the flat axal pencils (lines) that are polar to the generating lineoids of ν_i' with respect to (1). Denote by ν_i'' the plane common to *any* two of these pencils. The planes common to (1) and the special complex

$$\Omega(\nu'', \nu) = 0 \tag{3}$$

are identical with the planes of the polar flat axal pencils (lines) of the generating lineoids of ν_i''. The plane ν_i''' common to any two of these pencils is identical with ν_i'; for if L_1 and L_2 be the lineoids whose polar lines (pencils) give ν_i'', then, as the polar line l of any lineoid L of ν_i'' must lie in both L_1 and L_2, it follows that ν_i' is the locus of such polar lines l. We have accordingly the proposition: *Any pair of complexes C and C', of which one of them as C' is special, determines a third special complex C'' such that the assemblage of planes common to C and C' is identical with the assemblage common to C and C''.*

The director planes of C' and C'' are evidently *conjugates* with respect to C. In order, therefore, that ν_i' and ν_i'' shall be conjugate planes with respect to (1),

it is necessary and sufficient that

$$\Sigma c_i \nu_i = \lambda_1 \Omega(\nu', \nu) + \lambda_2 \Omega(\nu'', \nu), \tag{4}$$

whence

$$\left.\begin{array}{ll} c_1 = \lambda_1 \nu_4' + \lambda_2 \nu_4'', & c_4 = \lambda_1 \nu_1' + \lambda_2 \nu_1'', \\ c_2 = \lambda_1 \nu_5' + \lambda_2 \nu_5'', & c_5 = \lambda_1 \nu_2' + \lambda_2 \nu_2'', \\ c_3 = \lambda_1 \nu_6' + \lambda_2 \nu_6'', & c_6 = \lambda_1 \nu_3' + \lambda_2 \nu_3'', \end{array}\right\} \tag{5}$$

for some value of the ratio $\lambda_1 : \lambda_2$; or, writing the complexes in the form

$$\Sigma \frac{\partial \Omega(c)}{\partial c_i} \cdot \frac{\partial \Omega(\nu)}{\partial \nu_i} = 0, \quad \Sigma \frac{\partial \Omega(\nu)}{\partial \nu_i} \nu_i' = 0, \quad \Sigma \frac{\partial \Omega(\nu)}{\partial \nu_i} \nu_i'' = 0, \tag{6}$$

the condition may be written

$$\frac{\partial \Omega(c)}{\partial c_i} = \lambda_1 \nu_i' + \lambda_2 \nu_i'', \qquad (i = 1, 2, \ldots, 6). \tag{7}$$

By the aid of the condition

$$\Omega\left(\frac{\partial \Omega(c)}{\partial c} - \lambda_1 \nu'\right) = 0 \tag{8}$$

that the ν_i'' shall be coordinates of a plane, we readily find

$$\lambda_1 = \Omega(c) : \Sigma c_i \nu_i', \tag{9}$$

whence the coordinates of the conjugate ν'' of ν_i' are given by

$$\lambda_2 \nu_i'' = \frac{\partial \Omega(c)}{\partial c_i} - \frac{\Omega(c)}{\Sigma c_i \nu_i'} \nu_i'. \tag{10}$$

Four cases may arise:

$$\Omega(c) \neq 0, \quad \Sigma c_i \nu_i' \neq 0, \tag{a}$$
$$\Omega(c) \neq 0, \quad \Sigma c_i \nu_i' = 0, \tag{b}$$
$$\Omega(c) = 0, \quad \Sigma c_i \nu_i' \neq 0, \tag{c}$$
$$\Omega(c) = 0, \quad \Sigma c_i \nu_i' = 0. \tag{d}$$

In (a), which is the general case, the complex (1) is non-special and the plane ν_i' does not belong to (1). From the symmetry of (7), in respect to ν_i' and ν_i'', it appears that ν_i'' does not belong to (1). Hence, *of two conjugates with respect to a complex either both belong or neither belongs to the complex.* Equation (10) shows that under (a) to two planes ν_i' there correspond two planes ν_i'' and reciprocally (cf. §18). The equation

$$\Omega(\nu', \nu'') = 0 \tag{11}$$

signifies indifferently that ν_i'' belongs to (2) or that ν_i' belongs to (3); the director plane of a special complex may be considered as belonging to that complex; hence, if (11) be true, both ν_i' and ν_i'' belong to both (2) and (3) and hence also to (1), but this is contrary to (a). Hence, *two conjugates that do not belong to the given complex are non-collineoidal.*

In (b), ν_i' belongs to (1), which is non-special; $\lambda_1 = \infty$, and we have from (7)

$$\lambda_1 : \lambda_2 = -\nu_i'' : \nu_i',$$

which shows that every plane of (1) is self-conjugate.

In (c) the complex (1) is special, $\lambda_1 = 0$, and

$$\lambda_2 \nu_i'' = \frac{\partial \Omega(c)}{\partial c_i};$$

hence *the conjugate of any ν_i' with respect to a special complex not containing ν_i' is the director plane of the complex.*

In case (d), λ_1 is indeterminate; the meaning is that the conjugate of any ν_i' with respect to a special complex containing ν_i' is indeterminate, i. e., may be indifferently taken to be either ν_i' itself, as in (b), or the director plane, as in (c). One and the same line l is polar to all the lineoids of a given ν_i, and l is the intersection of ν_i and the director plane.

The results under the four cases may be summarized thus: *Given a complex C and let π stand for plane; π is self-conjugate or not so according as it belongs or does not belong to C; if π_1 and π_2 be any two planes, their conjugates are distinct or not according as C is non-special or special. In case of C special, the director plane is conjugate with all planes, itself included.*

IV.—*Linear Congruences of Planes, and Pencils of Complexes.*

20. A two-parameter system of planes, i. e., a system in which a plane has two degrees of freedom, has been named *congruence* (§11). It follows that the assemblage of planes that are common to two complexes is a congruence. The two-dimensional assemblage of planes constituting an axal pencil is a congruence. In like manner, the manifold of planes contained in a lineoid constitute a congruence, i. e., a bundle of planes is a congruence. A congruence being given, the number of planes it has in common with an arbitrary axal pencil will be called its *order*, while the number it has in common with an arbitrary bundle will be

called its *class.* In other words, the terms order and class of a congruence signify respectively the number of planes common to the congruence and an arbitrary line and the number common to the congruence and an arbitrary lineoid. The notions *order* and *class* of a plane congruence are seen to correspond respectively to *class* and *order* of a line congruence in the Plücker theory, the order of a line congruence being the number of its lines that go through an arbitrary point, and its class the number that lie in an arbitrary plane. A bundle of planes is of class one and order zero, while an axal pencil is of class zero and order one, just as in the Plücker theory a plane of lines is of class one and order zero, while a sheaf of lines is of order one and class zero.

21. We shall be chiefly concerned in this chapter with such congruences as are definable by two linear complexes of planes. Such congruences may be themselves called *linear.* We have seen that, given a linear complex, an arbitrary lineoid contains a flat axal pencil of planes belonging to the complex. It follows that an arbitrary lineoid contains one and but one plane of a given linear congruence. Reciprocally, an arbitrary line is contained in one and but one plane of the given congruence. It thus appears that a congruence composed of the planes common to two linear complexes is of order one and class one.

The assemblage of complexes represented by the equation

$$\lambda\Sigma c_i\nu_i + \lambda'\Sigma c_i'\nu_i \equiv \Sigma\,(\lambda c_i + \lambda' c_i')\,\nu_i = 0, \tag{1}$$

the c's being supposed given, and the λ's being parameters, will be called a *pencil* of complexes. The given complexes c_i and c_i', which determines it, may be called the *fundamental* complexes of the pencil.

If $\lambda_1 : \lambda_1'$ and $\lambda_2 : \lambda_2'$ be any two complexes of the pencil (1), this last is identical with the pencil

$$\Sigma\,[(\sigma\lambda_1 + \sigma'\lambda_2)\,c_i + (\sigma\lambda_1' + \sigma'\lambda_2')\,c_i']\,\nu_i = 0,$$

the σ's being parameters, for, in order to identify any given complex of either pencil with one of the other, we need only the relation

$$\lambda_1 : \lambda_2' = \sigma\lambda_1 + \sigma'\lambda_2 : \sigma\lambda_1' + \sigma'\lambda_2',$$

which, it is plain, can always be found. Hence, *the pencil of complexes determined by any two complexes of a given pencil is identical with the given pencil.*

22. It is obvious that the congruence defined by any two complexes c_i and c_i' is common to all the complexes of the pencil determined by c_i and c_i', and from the foregoing theorem it follows that the congruence may be regarded at will as defined by any pair whatever of the complexes of the pencil. A pencil of complexes and the congruence determined by a pair of the complexes may be said to *correspond.* In this way, to every pencil there corresponds a congruence, and conversely.

The condition that a complex of the pencil determined by the complexes c_i and c_i' shall be *special* is

$$\Omega(\lambda c + \lambda' c') \equiv \Omega(c)\lambda^2 + \Omega(c')\lambda'^2 + 2\Omega(c, c')\lambda\lambda' = 0. \tag{3}$$

Two cases are to be considered according as the roots of this equation, regarded as an equation in $\lambda : \lambda'$, are

(a) determinate

or (b) indeterminate.

Under (a), the general case, there fall two sub-cases:

(a′) $\Delta(c, c') \neq 0$,

(a″) $\Delta(c, c') = 0$,

where $$\Delta(c, c') \equiv \Omega(c)\,\Omega(c') - \Omega^2(c, c').$$

In (a) equation (3) has two and but two roots. These, which may be real or conjugate imaginaries, are definite and distinct. The pencil accordingly contains *two definite and distinct special complexes.* The congruence corresponding to the pencil consists of the planes that are at the same time collineoidal with the two director planes of the special complexes. These director planes may therefore be called the director planes of the congruence.

Denote by ν_i' and ν_i'' the director planes in question. The special complexes of the pencil

$$\Sigma(\lambda c_i + \lambda' c_i')\,\nu_i = 0 \tag{4}$$

will be given by the equations

$$\Omega(\nu', \nu) = 0, \quad \Omega(\nu'', \nu) = 0. \tag{5}$$

As the director plane of a special complex is, with respect to that complex, conjugate to all planes whatever, the planes ν_i' and ν_i'' are conjugate with respect

to both complexes (5). These two planes are conjugate (§19, Eq. 7) with respect to all and only the complexes of the pencil

$$\Sigma\,(\rho w_i' + \rho' w_i'')\,\nu_i = 0 \tag{6}$$

where

$$\begin{cases} w_1',\ w_2',\ w_3',\ w_4',\ w_5',\ w_6' = \nu_4',\ \nu_5',\ \nu_6',\ \nu_1',\ \nu_2',\ \nu_3', \\ w_1'',\ w_2'',\ \ldots\ldots\ldots\ldots = \nu_4'',\ \nu_5'',\ \ldots\ldots\ldots\ldots \end{cases}$$

Therefore (6) contains the complexes (5), and as these are also contained in (4), it follows that (4) and (6) are identical. Hence, *the two director planes of a linear congruence are conjugate with respect to every complex of the corresponding pencil.* Also *if two planes are conjugate with respect to each of two complexes, they are the director planes of the common congruence of those complexes.*

In sub-case (a) the two roots of (3) are equal, the two director planes *coincide,* are one plane. This plane is to be counted twice, once as belonging to the one and once as belonging to the other, of the two (coincident) special complexes. This double plane ($\nu_i' \equiv \nu_i''$) is, therefore, a plane of the congruence. All planes of the congruence are collineoidal with ν_i', but the converse is not true, for while a plane of a congruence has two, a plane merely required to be collineoidal with a given plane has three, degrees of freedom. The precise way in which a plane having three degrees of freedom as specified, loses one degree on being required to belong to a linear congruence having the given plane ν_i' for double directrix, may be seen as follows: If π belongs to the congruence, the lineoid $(\pi,\ \nu_i')$, since it contains two planes of each complex of the pencil, is, in regard to each complex, the polar lineoid of the line $(\pi,\ \nu_i')$. Of the ∞^2 planes containing this line, only ∞^1 planes, viz., those contained in the lineoid $(\pi,\ \nu_i')$, belong to the congruence. If, therefore, a bilinear relation be set up between the generating lines and the generating lineoids of any plane ν_i', then the assemblage of planes obtained by taking all and only flat axal pencils that are contained in the lineoids corresponding to the axes (lines) is a linear congruence having ν_i' as double directrix.

The necessary and sufficient conditions for the occurrence of case (b) are

$$\Omega\,(c) = 0, \quad \Omega\,(c') = 0, \quad \Omega\,(c,\ c') = 0.$$

The first two of these equations indicate that the fundamental complexes c_i and c_i' are special, and the third equation signifies that the director planes of c_i and c_i' are collineoidal. The indetermination of $\lambda : \lambda'$ shows that the pencil is com-

posed of special complexes. The corresponding congruence has for director planes *any* pair of an infinity of planes, the director planes of the pencil, all of which belong to the congruence. We may readily prove that *these director planes constitute a flat axal pencil.* For if

$$\Sigma(\lambda c_i + \lambda' c')\,\nu_i = 0$$

be any complex of the pencil of complexes and if ν_i' be the corresponding director plane, then

$$\nu_i' = \frac{\partial\Omega(\lambda' c + \lambda' c')}{\partial(\lambda c_i + \lambda' c_i')}, \qquad (i = 1, \ldots, 6)$$

or

$$\nu_i' = \lambda\frac{\partial\Omega(c)}{\partial c_i} + \lambda'\frac{\partial\Omega(c')}{\partial c_i'},$$

which shows that as $\lambda:\lambda'$ varies through all real values, the director plane ν_i' generates a flat axal pencil p, namely, that determined by the director planes of the complexes c_i and c_i'. The congruence consists of all the planes of which each is collineoidal with each plane of p. It is accordingly composed of two hyperpencils, one of each kind: the *bundle* contained in the lineoid containing the planes of p, and the *axal pencil* having for its axis the axis of p. Each of these components is of itself a congruence, the former of order zero and class one, the latter of order one and class zero, showing what should be the case that the congruence under consideration is of order one and class one.

23. The propositions of line geometry that correspond to the foregoing may be briefly stated as follows:

Case (a), sub-case (a′). The linear congruence has two distinct non-intersecting directrices (lines). The assemblage of lines joining the latter is the congruence. The directrices are conjugate lines with respect to every line complex of the pencil of complexes defining the congruence.

Case (a), sub-case (a″). The congruence has one (double) directrix. If a bilinear relation be established between the points and the planes of any given line l, this line is the double directrix of the congruence obtained by taking all and only the lines of such flat pencils as lie each in the plane that corresponds to the vertex (a point of l) of the pencil.

Case (b). The congruence has an infinity of directrices, which belong to it and which constitute a flat pencil of lines. The congruence is composed of two congruences: the assemblage of lines of the plane of the pencil and the sheaf

of lines containing the vertex of the pencil, the former component being of order zero and class one, while the latter is of order one and class zero.

24. The geometric properties of the discriminant Δ of $\Omega(\lambda c + \lambda' c')$ show that Δ is an invariant both under a reversible linear transformation of the variables v and under such a transformation of the λ's; for the coincidence or non-coincidence of the director planes of a congruence is independent alike of the coordinates employed and of the particular choice of a pair out of the corresponding pencil of complexes for fundamental complexes. We will merely state without proof that, if M_1 be the modulus of the v-transformation and Δ_1 be the new discriminant, and if M_2 be the modulus of the λ-transformation and Δ_2 the resulting discriminant, then

$$\begin{cases} \Delta_1 = M_1^4 \Delta, \\ \Delta_2 = M_2^2 \Delta. \end{cases}$$

The λ-transformation being merely equivalent to replacing the fundamental complexes c_i and c_i' by a new pair as $\lambda_1 c_i + \lambda_1' c_i'$ and $\lambda_2 c_i + \lambda_2' c_i'$, we may write

$$\Delta_2 = \Delta(\lambda_1 c + \lambda_1' c', \lambda_2 c + \lambda_2' c') = (\lambda_1 \lambda_2' - \lambda_2 \lambda_1')^2 . \Delta.$$

As the vanishing or non-vanishing of Δ is not a mark of any particular pair of complexes of the pencil but characterizes the pencil as such, we may name $\Delta(c_1 c')$ the *discriminant of the pencil* $\lambda c_i + \lambda' c_i'$ *or of the corresponding congruence.* A pencil of complexes being given, it contains two distinct or two coincident special complexes, and the corresponding congruence has two distinct or two coincident director planes, according as the discriminant vanishes or does not vanish.

V.—*Projective Transformations by Means of Complexes. Orthogonal Complexes: Involution.*

25. We have seen that a linear complex of planes serves as a means of dualistic transformation according to which lines or lineoids correspond to lineoids or lines, and planes correspond to planes. If c_i be any linear complex and if π and π' be any pair of conjugate planes with respect to c_i, then the generating lines or lineoids of π or π' are projectively related through c_i to the generating lineoids or lines of π' or π, any line or lineoid being with respect to c_i the polar of the corresponding lineoid or line. If l_1, l_2, l_3, l_4 be any four lines of π or π',

and L_1, L_2, L_3, L_4 be the corresponding lineoids (of π' or π), the anharmonic ratio of the l's is equal to that of the L's, i. e.,

$$(l_1 l_2 l_3 l_4) = (L_1 L_2 L_3 L_4).$$

If π belong to c_i, π is self-conjugate, and if generated by l it will at the same time be generated by L, the (polar) correspondent of l. The plane being at once a locus of lines (a pencil of lines) and an envelope of lineoids (a pencil of lineoids), we have the proposition: *Any linear complex of planes establishes a projective correspondence between the lines and the lineoids of each of its planes.*

If now we suppose π to be common to the two complexes c_i and c_i', and if to the lineoids L_1, L_2, L_3, L_4 of π there correspond with reference to c_i, the lines l_1, l_2, l_3, l_4 and with reference to c_i' the lines l_1', l_2', l_3', l_4', then, as

$$(L_1 L_2 L_3 L_4) = (l_1 l_2 l_3 l_4),$$
$$(L_1 L_2 L_3 L_4) = (l_1' l_2' l_3' l_4'),$$

we have
$$(l_1 l_2 l_3 l_4) = (l_1' l_2' l_3' l_4').$$

Accordingly, if we regard π as *two superposed* pencils, viz., of lines l (associated with c_i) and of (the same) lines l' (associated with c_i'), it is seen that these pencils are brought into projective relation by means of the complexes c_i and c_i'. Reciprocally, π may be conceived as two superposed pencils of (its generating) lineoids, L and L', and these, too, are projectively related through the complexes in question. Given either of the line (or lineoid) pencils, the other pencil (of the same kind) is obtainable from the given one by means of a linear line (or lineoid) transformation of π. Hence, *the assemblage of two given complexes plays the rôle of a definite linear transformation at the same time of the lines and the lineoids of any given plane of the corresponding congruence.*

As π is common to all the complexes of the pencil

$$\Sigma\,(\lambda c_i + \lambda' c_i')\,\nu_i = 0\,, \tag{7}$$

it follows that the lines and lineoids of π are transformed by every pair, $\lambda' : \lambda = k_1$, $\lambda' : \lambda = k_2$, of these complexes. The equation of the transformation will assume its simplest form when referred to its *foci*. What then are these? Denote by π_d and π_d' the director planes of the congruence, and let l_d and l_d' be the lines and L_d and L_d' the lineoids determined by π and the director planes. As π_d and π_d' are conjugates with respect to both k_1 and k_2, l_d and L_d' are each the other's polar in regard to both k's. Similarly, l_d' and L_d. Hence l_d and l_d' are the foci

of the line transformation, and L_d and L'_d are the foci of the lineoid transformation, effected on π by the pair k_1, k_2 of complexes. Taking l_d and l'_d as base elements of homogeneous coordinates, $z = x_1 : x_2$, of the lines of the pencil determined by l_d and l'_d, the line transformation assumes the form

$$z' = \rho z.$$

In like manner, if z be interpreted as coordinates of the lineoids of the pencil lineoids referred to L_d and L'_d as base elements, the lineoid transformation takes the form

$$z' = \rho' z.$$

26. In case of the line transformation, the *characteristic* constant ρ is the anharmonic ratio formed by any pair z and z' of corresponding lines of π with the foci l_d and l'_d. Writing

$$\alpha = \frac{1}{2i} \log \rho,$$

α will be the (generalized) angle of the lines z and z'. If the foci be the isotropic lines of the pencil in question, α will be the ordinary angle of z and z'. In like manner α', where

$$\alpha' = \frac{1}{2i} \log \rho'$$

is the angle between any pair z and z' of corresponding lineoids.

42. Since the transformation is determined completely by the given complexes k_1 and k_2, it must be possible to express ρ and ρ' in terms of the parameters k_1, k_2 and the coefficients of the fundamental complexes c_i and c'_i. We proceed to determine such expressions for ρ.

Let L be any lineoid of π. With respect to each complex of the pencil (7), L has a polar line l. By virtue of this one-to-one correspondence, if k_1, k'_1, k_2, k'_2 be any four of the complexes and l_1, l_2, l_3, l_4 the corresponding polar lines of L, we shall have

$$(l_1 \, l'_1 \, l_2 \, l'_2) = (k_1 \, k'_1 \, k_2 \, k'_2).$$

If, now, we replace k'_1 and k'_2 respectively by s_1 and s_2, where $\lambda' : \lambda = s_1$, $\lambda' : \lambda = s_2$ are the *special* complexes of (7), l'_1 and l'_2 will be replaced by the foci l_d and l'_d and we shall obtain, since l_1 and l_2 are any pair of correspondents,

$$\rho = (l_1 \, l_d \, l_2 \, l'_d) = (K_1 \, S_1 \, K_2 \, S_2).$$

In precisely like manner, it may be shown that

$$\rho' = (k_1 s_1 k_2 s_2)\,,$$

whence

$$\rho = \rho', \quad \alpha = \alpha'.$$

Hence, *The line and lineoid transformation effected by two given complexes* k_1, k_2 *on any plane* π *of their common congruence are algebraically identical. The transformation remains unchanged if* π *be supposed to generate the congruence. The angle of two corresponding lines of a given* π *is constant. The same is true of the angle of two corresponding lineoids.*

27. The angle of any pair of lines of one plane is equal to that of any pair of any other plane of the congruence. The two pairs are, however, not congruent (in the ordinary sense). The equality is merely arithmetic, not geometric. The systems of measurement in the one plane and in the other are not the same. The foci (the Cayleyan *absolute*) in the one plane are not congruent with the foci in case of the other. Like remarks apply to the lineoid transformations, and to the equation $\alpha = \alpha'$.

28. There is a simple infinity, ∞^1, of linear transformations having the same assigned foci. On the other hand, there are ∞^2 pairs of complexes defining one and the same congruence and each pair gives a line (lineoid) transformation of the planes. It is, therefore, to be expected that, if one pair gives a transformation, there are ∞^1 pairs giving the same transformation. That such is the case appears in the equation

$$\rho = (k_1 s_1 k_2 s_2)\,,$$

which, being linear in k_1 and in k_2, is satisfied by ∞^1 of pairs of values of the k's This statement is independent of the value of ρ, and as the foci in case of any plane are the same for all pairs of k's, we have the proposition: *The lines (lineoids) of every plane of a congruence are transformed by* ∞^1 *distinct transformations by pairs of complexes of the corresponding pencil of complexes, each transformation is effected by* ∞^1 *pairs, and all transformations of any given plane have the same foci*

Holding ρ fixed, the above equation may be regarded as a *linear transformation of the pencil* (7) *of complexes.* The foci of the transformation are the special complexes s_1 and s_2, for on writing

$$\rho = \frac{(k_1 - s_1)\,(k_2 - s_2)}{(s_1 - k_2)\,(s_2 - k_1)}\,,$$

it is seen that k_1 and k_2 coincide when and only when one of them coincides with s_1 or s_2 (the s's are here supposed distinct). The constant ρ is, then, the anharmonic ratio formed by *any* pair k_1 and k_2 of complexes with the pair of special complexes s_1 and s_2. We may therefore call

$$\alpha \equiv \frac{1}{2i} \log \rho,$$

the *angle of the complexes* k_1 *and* k_2. Hence the proposition: *The angle of any pair of complexes is equal to the angle between any two corresponding lines* (*lineoids*) *in the transformation effected by the given complexes upon any plane of their common congruence.*

The case where $\rho = -1$ and $\alpha = \frac{\pi}{2}$ is of special interest. In this case the complexes k_1 ank k_2 may be said to be *orthogonal*, or, since the pair k_1, k_2 is *harmonic* to the pair s_1, s_2, the complexes k_1 and k_2 may be said to be *in involution* with respect to the two special complexes. The condition for such orthogonality or involution is

$$2k_1 k_2 - (s_1 + s_2)(k_1 + k_2) + 2s_1 s_2 = 0, \tag{8}$$

which shows that *every complex of a pencil of complexes is in involution with one and* (*in general*) *only one other complex of the pencil.*

It is plain also that *any pair of corresponding lines* (*lineoids*) *of a plane transformed by two orthogonal complexes is harmonic to the pair of foci in that plane.*

If, in equation (8), we let $k_1 = 0$, then $k_2 = 2s_1 s_2 : (s_1 + s_2)$, and this will be ∞ when and only when $s_1 + s_2 = 0$, or, since the s's are the roots of equation (3), the condition is

$$\Omega(c, c') = 0.$$

The values $k_1 = 0$, $k_2 = \infty$ correspond to the fundamental complexes c_i and c'_i of the pencil (7). The vanishing of $\Omega(c, c')$ is, therefore, the condition that the complexes c_i and c'_i shall be orthogonal. This function of the coefficients c is that which, in the Plücker geometry, Klein has called the *simultaneous invariant* of the complexes c_i and c'_i.

29. Suppose that the plane ν'_i belongs to c'_i, and that ν''_i is the conjugate of ν'_i with respect to c_i, then (IV, Eq. 7),

$$\frac{\partial \Omega(c)}{\partial c_i} = \lambda_1 \nu'_i + \lambda_2 \nu''_i,$$

whence
$$\Sigma \frac{\partial \Omega(c)}{\partial c_i} c_i' = \lambda_1 \Sigma c_i' \nu_i' + \lambda_2 \Sigma c_i' \nu_i'',$$
i. e.,
$$2\Omega(c, c') = \lambda_2 \Sigma c_i' \nu_i''.$$
Since, by hypothesis,
$$\Sigma c_i' \nu_i' = 0.$$

Accordingly, if ν_i'' belongs to c_i', the simultaneous invariant vanishes. Therefore, *if two complexes are such that one of them contains a plane conjugate to one of its planes with respect to the other complex, the complexes are orthogonal.*

It is, moreover, seen that, if ν_i'' is in c_i', then ν_i'''' the conjugate of any other plane ν_i''' of c_i' is also in c_i', i. e., the planes of c_i' fall into pairs of conjugates with respect to c_i, and, as the relation is a reciprocal one, the planes of c_i fall into pairs of conjugates with respect to c_i. Two complexes thus related may be said to be each its own conjugate, or *self-conjugate*, or self-polar, with respect to the other. *This property of self-conjugateness characterizes both complexes of every orthogonal pair and might be taken as the defining property of orthogonality.*

30. The notion of orthogonality has as yet been attached only to pairs of non-special complexes. Nevertheless if (following Klein and Koenigs in the line theory) we agree to say that two complexes are orthogonal *always* when their simultaneous invariant is zero, then it readily follows: (1) *That every special complex is orthogonal to all complexes containing its director plane, and conversely;* (2) *that the necessary and sufficient condition for the orthogonality of two special complexes is that their director planes be collineoidal.*

Consideration of the *net*
$$\lambda_0 \Sigma c_i \nu_i + \lambda_1 \Sigma c_i' \nu_i + \lambda_2 \Sigma c_i'' \nu_i = 0,$$
and the *web*
$$\lambda_0 \Sigma c_i \nu_i + \cdots\cdots + \lambda_3 \Sigma c_i''' \nu_i = 0,$$
of complexes, which correspond respectively to the one-parameter system (configuration, plane-series) and the zero-parameter system of planes, is reserved.

COLUMBIA UNIVERSITY, *Aug.* 10, 1901.